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## The general structure of eigenvalues in nonlinear oscillators

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**Abstract.** Hilbert spaces of bounded one-dimensional nonlinear oscillators are studied. It is shown that the eigenvalue structure of all such oscillators have the same general form. They depend only on the ground state energy of the system and a single function  $\lambda(H)$  of the Hamiltonian operator  $H$ . It is also found that the Hilbert space of the nonlinear oscillator is unitarily *inequivalent* to the Hilbert space of the simple harmonic oscillator, providing an explicit example of Haag's theorem. A number operator for the nonlinear oscillator is constructed and the general form of the partition function and average energy of a nonlinear oscillator in contact with a heat bath is determined. Connection with the WKB result in the semiclassical limit is made. The analysis is then applied to the case of the  $x^4$  anharmonic oscillator as an explicit example.

### 1. Introduction

In this paper we study the general structure of the energy eigenvalues for one-dimensional nonlinear oscillators. To be specific, we are interested in Hamiltonians which have the form,

$$H = (a^\dagger a + \frac{1}{2}I) + V(a, a^\dagger) \quad (1)$$

where  $V(a, a^\dagger)$  is the interaction Hamiltonian and is a function of  $a$  and  $a^\dagger$ , the creation and annihilation operators for the simple harmonic oscillator (SHO), and  $I$  is the identity operator. (We have set the SHO energy scale to '1'.) We shall restrict ourselves to bounding potentials for which  $V \rightarrow \infty$  when  $|x| \rightarrow \infty$ . When  $V$  is a polynomial in  $a$  and  $a^\dagger$  consisting of terms  $(a^\dagger)^r a^s$ , the degree  $l$  of  $V$  is the maximum value of  $l = r + s$  for the polynomial. As is well known, when  $l \leq 2$  the Hamiltonian is easily diagonalizable by either shifting  $a$  by a constant (for  $l = 1$ ) or by a Bogoluibov transformation (for  $l = 2$ ). Nonetheless, the results of these analyses have had far-reaching applications, such as coherent and squeezed quantum states [1] in quantum optics, and the theories of superfluidity and superconductivity [2].

When  $l > 2$  the oscillator is nonlinear with the classical example being the anharmonic oscillator with a  $x^4$  or  $(a + a^\dagger)^4$  interaction potential. This particular nonlinear oscillator has been extensively studied since the early 1970s ([3–6]; see [7] for a review of the literature), due mainly to its equivalence to the one-dimensional  $\phi^4$  quantum field theory. It was hoped that a detailed study of this simplified system would shed some light on the structure of the  $\phi^4$  theory in higher dimensions. Research on this oscillator continues today, mainly because it provides a natural test bed for such approximation schemes as the strong coupling expansion [8], modified perturbation schemes [9, 10], variational modified perturbation theories [11], lattice methods [12], etc. More recently, Bender and Bettencourt [13] have provided a deeper understanding of the system by using multiple-scale perturbation theory to show that the frequency of oscillation depends on the energy of the state. This was interpreted by them as an operator form of mass renormalization.

The main purpose of this paper is not to present a new method of calculating the energy eigenvalues of nonlinear oscillators, although we shall end up doing so. Rather, it is to study the general structure of both the Hilbert space and corresponding eigenvalues for oscillators with arbitrary binding  $V(\mathbf{a}, \mathbf{a}^\dagger)$ . The approach we shall take most closely follows the analysis done for the  $l \leq 2$  oscillators. Namely, we shall construct, in much the same way, operators  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{a}}^\dagger$  from  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  which diagonalizes the Hamiltonian. We find that unlike the SHO operators,  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{a}}^\dagger$  obey the commutation relation  $[\tilde{\mathbf{a}}, \tilde{\mathbf{a}}^\dagger] = \lambda(\mathbf{H})$  where in general  $\lambda(\mathbf{H})$  is a function of  $\mathbf{H}$  and is an operator. Its precise form depends on the specific choice of  $\mathbf{H}$  and is a constant only when  $l \leq 2$ . The study of any nonlinear oscillator thereby reduces to the study of operators having this commutation relation along with the determination of  $\lambda(\mathbf{H})$  and the ground state energy of the system. Because  $\lambda$  is not a constant function in general, we find that  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{a}}^\dagger$  cannot be unitarily equivalent to  $\mathbf{a}$  and  $\mathbf{a}^\dagger$ . Only in the special case when  $l \leq 2$  does such a transformation exist. Consequently, the Hilbert space of the nonlinear oscillator is generally *unitarily inequivalent* to that of the SHO. This is an explicit example of Haag's theorem, first proposed by Haag in 1955 [14] (see also [15]) for quantum field theories. Haag actually proved a weaker version of the theorem by showing that the unitary transformation between the non-interacting and interacting quantum field theories via the interaction picture does not exist. Later, this result was extended by Hall and Wightman [16] (see also [17]) who showed that based on the Wightman axioms, the expectation values of the product of four or fewer fields of an interacting theory is unitarily inequivalent to those of the free theory.

There have been other approaches in using algebraic methods to analyse nonlinear oscillators, of course, such as the action angle or time operator methods (see, for example, [18–22]). However, both of these methods are generalizations of classical analytical techniques to quantum mechanical systems. They rely on the existence of phase  $\Phi$  and time  $T$  operators which are canonical to the SHO number and Hamiltonian operators:  $[N, \Phi] = I, [H, T] = I$ . Because of the positivity of the spectrum of both  $N$  and  $H$  for bound systems, such operators do not exist in the usual quantum mechanical system [23] (see, however, [24–26] for the existence of such operators in extended quantum mechanical systems). In this sense these methods of solution are 'formal'. The approach we have taken in this paper does not suffer from these problems. It is not a generalization of classical techniques but is instead a generalization of the Bogolubov transformation and is inherently quantum mechanical in nature. Classical solution techniques are used only in the semiclassical limit where they are expected to be valid.

The rest of this paper is organized as follows. In section 2 the general Hilbert space and energy eigenvalue structure of nonlinear oscillators are analysed. It is found that they both depend on a function  $\lambda(\mathbf{H})$  of the Hamiltonian. A number operator is constructed and the Heisenberg equations of motion are solved. The utility of this analysis is shown in section 3 where it is applied to thermal or KMS states and general statements on the structure of the quantum statistical mechanics of nonlinear oscillators will be made. This also serves as an independent check of our methodology and results. Indeed, it is shown that both the partition function  $Z$  and the average energy  $\langle \mathbf{H} \rangle_T$  for nonlinear oscillators are similar in form to those of the SHO. In section 4 a method of explicitly constructing  $\lambda, \tilde{\mathbf{a}}$  and  $\tilde{\mathbf{a}}^\dagger$  from  $\mathbf{H}, \mathbf{a}$  and  $\mathbf{a}^\dagger$  is outlined and in section 5 a connection between this method and the semiclassical WKB result is shown. It is thereby demonstrated that in the semiclassical limit  $\lambda, \tilde{\mathbf{a}}$  and  $\tilde{\mathbf{a}}^\dagger$  exist. We then apply this construction to the  $x^4$  potential in section 6 and reproduce the usual perturbative result to second order as well as the WKB result in the semiclassical limit. This thereby provides an explicit example of the appropriateness of this approach. Concluding remarks are given in section 7.

A note first on notation and convention: operators shall be denoted by boldfaced letters and we are interested in nonlinear mappings of these operators. In particular, we are interested in

mappings of Hermitian operators  $\mathbf{K}$  such as  $f(\mathbf{K})$ , by which we mean the following. Consider a function defined on the real line  $\mathcal{R}$  which is expandable in the series

$$f(K) = \sum_{n=0}^{\infty} f_n K^n \quad (2)$$

where  $K$  is a real number. If the radius of convergence of this series contains the spectrum of  $\mathbf{K}$ , then  $f(\mathbf{K})$  is well defined and taken to be

$$f(\mathbf{K}) \equiv \sum_{n=0}^{\infty} f_n \mathbf{K}^n. \quad (3)$$

When we write  $f(\mathbf{K})$  it is understood that  $f(\mathbf{K})$  is an *operator* while when we write  $f(K)$  we are considering it as a c-number function only. Operationally,  $f(\mathbf{K})$  can be obtained from  $f(K)$  by the replacement  $K \rightarrow \mathbf{K}$ . If we are referring to the functional form of the function we shall denote it by  $f$  only.

## 2. General structure

We construct eigenstates of nonlinear oscillators by considering nonlinear mappings of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$ . In particular, given a Hamiltonian  $\mathbf{H}$  constructed from  $\mathbf{a}$ , and  $\mathbf{a}^\dagger$ , we seek solutions of the operator equation

$$[\tilde{\mathbf{a}}, \mathbf{H}] = \lambda(\mathbf{H})\tilde{\mathbf{a}} \quad (4)$$

where  $\tilde{\mathbf{a}}$  is understood to be a function of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$ . This is an eigenvalue equation with  $\tilde{\mathbf{a}}$  being the ‘eigenoperator’ of  $\mathbf{H}$  and  $\lambda(\mathbf{H})$  its corresponding ‘left eigenvalue’, although unlike the standard eigenvalue equation  $\lambda(\mathbf{H})$  is a function of  $\mathbf{H}$  and the ordering in (4) is important<sup>†</sup>. Equation (4) does not determine  $\tilde{\mathbf{a}}$  uniquely since if  $\tilde{\mathbf{a}}$  satisfies (4), then so does  $g(\mathbf{H})\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{a}}g(\mathbf{H})$  where  $g$  is any function of  $\mathbf{H}$ . A normalization for  $\tilde{\mathbf{a}}$  is needed, which we choose to be

$$\mathbf{H} = \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} + e_g \mathbf{I} \quad (5)$$

since it diagonalizes the Hamiltonian explicitly.  $e_g$  is the ground state energy of the system which also needs to be determined. (This is very similar to the way one determines the Bogoluibov transformation which diagonalizes the  $l = 2$  Hamiltonian  $\mathbf{H} = \mathbf{a}^\dagger \mathbf{a} + i\epsilon(\mathbf{a}^2 - (\mathbf{a}^\dagger)^2)/2$ , but now  $\lambda(\mathbf{H})$  is a function of  $\mathbf{H}$ .) Using this normalization equation (5) in equation (4) we find that

$$[\tilde{\mathbf{a}}, \tilde{\mathbf{a}}^\dagger] = \lambda(\mathbf{H}). \quad (6)$$

To show that  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{a}}^\dagger$  creates and annihilates eigenstates of  $\mathbf{H}$ , we make use of the identity,

$$[\tilde{\mathbf{a}}, \mathbf{H}^n] = \{(\lambda(\mathbf{H}) + \mathbf{H})^n - \mathbf{H}^n\}\tilde{\mathbf{a}} \quad (7)$$

obtained using (4). Then for any given function  $M(\mathbf{H})$  which is expandable in a power series,

$$[\tilde{\mathbf{a}}, M(\mathbf{H})] = \{M(\lambda(\mathbf{H}) + \mathbf{H}) - M(\mathbf{H})\}\tilde{\mathbf{a}}. \quad (8)$$

With this we see that if  $\tilde{\mathbf{a}}$  is an eigenoperator of  $\mathbf{H}$  with left eigenvalue  $\lambda(\mathbf{H})$ , then so is  $\tilde{\mathbf{a}}^n$ ,

$$[\tilde{\mathbf{a}}^n, \mathbf{H}] = \{\lambda(\mathbf{H}) + \lambda(\lambda(\mathbf{H}) + \mathbf{H}) + \dots + \lambda(\lambda(\dots \lambda(\lambda(\mathbf{H}) + \mathbf{H}) \dots + \mathbf{H}) + \mathbf{H})\}\tilde{\mathbf{a}}^n. \quad (9)$$

<sup>†</sup> This is in contrast to the operator equation  $[\tilde{\mathbf{a}}, \mathbf{H}] = \tilde{\mathbf{a}}\lambda^R(\mathbf{H})$  for which  $\lambda^R(\mathbf{H})$  is the ‘right eigenvalue’. In general,  $\lambda \neq \lambda^R$ . Indeed, while equations analogous to equations (7)–(9) would hold with the right eigenvalue solutions, equation (5) no longer implies an analogous equation (6) for the right eigenvalue. Moreover, construction of the Hilbert space would no longer follow the same method as given on page 3812.

Given equation (9), the Hilbert space  $\mathcal{H}_{nl}$  for nonlinear oscillators and their energy eigenvalues are now easily constructed in much the same way as the SHO Hilbert space  $\mathcal{H}_{SHO}$ . To do so, we first note the following. Let  $|A\rangle$  be an eigenstate of the operator  $\mathbf{A}$  with eigenvalue  $A$  so that  $\mathbf{A}|A\rangle = A|A\rangle$ . Next, consider a function  $\chi(x)$  which is expandable in a power series for all values of  $A$  so that

$$\chi(\mathbf{A}) = \sum_{n=0}^{\infty} \chi_n \mathbf{A}^n \quad (10)$$

is well defined. It is then clear that

$$\chi(\mathbf{A})|A\rangle = \sum_{n=0}^{\infty} \chi_n \mathbf{A}^n |A\rangle = \sum_{n=0}^{\infty} \chi_n A^n |A\rangle = \chi(A)|A\rangle \quad (11)$$

and  $|A\rangle$  is also an eigenstate of  $\chi(\mathbf{A})$  with eigenvalue  $\chi(A)$ .

Now consider the operator  $\hat{\mathbf{H}} = \mathbf{H} - e_g \mathbf{I} = \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}}$ . Then, if  $|\phi_0\rangle$  is an eigenstate of  $\hat{\mathbf{H}}$  with eigenvalue  $\phi_0$ ,

$$\phi_0 = \langle \phi_0 | \hat{\mathbf{H}} | \phi_0 \rangle = |\tilde{\mathbf{a}} | \phi_0 \rangle|^2 \geq 0 \quad (12)$$

and all the eigenvalues of  $\hat{\mathbf{H}}$  are non-negative. Next consider the state  $|\phi_{-1}\rangle \equiv \tilde{\mathbf{a}} | \phi_0 \rangle$ . Then

$$\hat{\mathbf{H}} | \phi_{-1} \rangle = \phi_0 | \phi_{-1} \rangle - \lambda(\mathbf{H}) | \phi_{-1} \rangle \quad (13)$$

or, equivalently,  $(\hat{\mathbf{H}} + \lambda(\mathbf{H})) | \phi_{-1} \rangle = \phi_0 | \phi_{-1} \rangle$ . From the above, since  $\lambda$  is a function of  $\hat{\mathbf{H}} + e_g \mathbf{I}$ ,  $|\phi_{-1}\rangle$  must also be an eigenstate of  $\hat{\mathbf{H}}$ . We shall label its eigenvalue as  $\phi_{-1}$ . Then equation (13) reduces to  $\phi_{-1} = \phi_0 - \lambda(\phi_{-1} + e_g)$ . If  $\lambda(e) > 0$  for all  $e$ , we have  $\phi_{-1} < \phi_0$  (see appendix A).

Similarly, the states  $|\phi_{-n}\rangle \equiv \tilde{\mathbf{a}}^n | \phi_0 \rangle$  are also eigenstates of  $\hat{\mathbf{H}}$  with eigenvalues  $\phi_{-n}$ . Moreover, for  $\lambda(e) > 0$  they satisfy a sequence of strict inequalities

$$\phi_{-n} < \phi_{-(n-1)} < \dots < \phi_{-1} < \phi_0. \quad (14)$$

Since  $\phi_{-n} \geq 0$  for all  $n$ , this sequence must end. Namely, for some  $m$ ,  $\phi_{-m} = 0$ . Then  $0 = \langle \phi_{-m} | \hat{\mathbf{H}} | \phi_{-m} \rangle = |\tilde{\mathbf{a}} | \phi_{-m} \rangle|^2$  or  $\tilde{\mathbf{a}} | \phi_{-m} \rangle = 0$ . The ground state is then identified as  $|\Omega\rangle = | \phi_{-m} \rangle$  and  $\mathbf{H} | \Omega \rangle = e_g | \Omega \rangle$ , thereby justifies our calling  $e_g$  the ground state energy<sup>†</sup>.

Thus, if  $\lambda(e) > 0$  for all  $e$ , a ground state exists and  $\mathcal{H}_{nl}$  is spanned by the states

$$|n\rangle_{nl} = \frac{(\tilde{\mathbf{a}}^\dagger)^n | \Omega \rangle_{nl}}{\sqrt{A_n}} \quad (15)$$

where

$$A_n = \lambda(e_g) \cdot (\lambda(e_g) + \lambda(\lambda(e_g) + e_g)) \cdot \dots \cdot (\lambda(e_g) + \lambda(\lambda(e_g) + e_g) + \dots + \lambda(\lambda(\dots \lambda(\lambda(e_g) + e_g) \cdot \dots + e_g) + e_g)). \quad (16)$$

They are eigenstates of  $\mathbf{H}$  with eigenvalues  $e_n$  where

$$e_n = e_{n-1} + \lambda(e_{n-1}) \\ = e_g + \lambda(e_g) + \lambda(\lambda(e_g) + e_g) + \dots + \lambda(\lambda(\dots \lambda(\lambda(e_g) + e_g) \cdot \dots + e_g) + e_g). \quad (17)$$

$\lambda$  thereby determines the splitting between successive energy levels.

If  $\lambda$  is the constant function, then from (17) we see that the energy levels of the oscillator are equally spaced. As we shall show in section 4, this is only possible for  $l \leq 2$ , which is well known. When  $l > 2$ ,  $\lambda(\mathbf{H})$  is a function of  $\mathbf{H}$  and this equal spacing no longer occurs.

<sup>†</sup> Aside from requiring that  $\lambda$  be a positive definite function, there are no other positivity requirements for (2) and (4). In appendix A, this positivity requirement is shown to hold for a wide variety of potentials in the semiclassical limit. This is in contrast to the commutation relations for the time and phase operators.

Note, however, that both equation (5) and the commutation relation (6) are invariant under unitary transformations:  $\tilde{a} \rightarrow U\tilde{a}U^\dagger$ . As usual, unitary transformations are canonical transformations which preserve the commutation relation. For the SHO,  $\lambda = 1$ , while for a nonlinear oscillator  $\lambda(\mathbf{H})$  is a function of  $\mathbf{H}$ . Since a unitary transformation cannot change the functional form of  $\lambda$ ,  $\tilde{a}$  and  $\mathbf{a}$  are unitarily inequivalent. *Consequently, the Hilbert spaces  $\mathcal{H}_{nl}$  and  $\mathcal{H}_{SHO}$  are unitarily inequivalent.*

It is well known that under certain conditions any two solutions of the *canonical commutation relation*  $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{I}$  are connected by a unitary transformation. Our result does not contradict this. The operators  $\tilde{a}$  and  $\tilde{a}^\dagger$  which create and annihilate eigenstates of a general one-dimensional oscillator do not in general obey this canonical commutation relation except in the special case of  $l \leq 2$ . Instead,  $[\tilde{a}, \tilde{a}^\dagger] = \lambda(\mathbf{H})$ . Indeed, it is precisely for this reason that  $\tilde{a}$  and  $\mathbf{a}$  cannot be related to one another by a unitary transformation.

If we now consider Hamiltonians of the form of equation (1) in which  $V(\mathbf{a}, \mathbf{a}^\dagger)$  is controlled by a single coupling constant  $\epsilon$ , we can label the Hilbert space for each  $\epsilon$  as  $\mathcal{H}_\epsilon$ . Then  $\mathcal{H}_\epsilon$  is unitarily inequivalent to  $\mathcal{H}_{SHO}$ . Moreover, there cannot be a unitary transformation which maps  $\mathcal{H}_\epsilon \rightarrow \mathcal{H}_{\epsilon'}$  when  $\epsilon \neq \epsilon'$ . If there were, then using a succession of these transformations we can construct a unitary transformation mapping  $\mathcal{H}_\epsilon$  to  $\mathcal{H}_{SHO}$  and the two Hilbert spaces would be unitarily equivalent. Thus for different values of  $\epsilon$  the Hilbert spaces  $\mathcal{H}_\epsilon$  are inequivalent to one another. This is a concrete example of Haag's theorem (see [15]), first proved for quantum field theories using Lorentz invariance.

Because the spectrum of  $\tilde{a}^\dagger\tilde{a}$  no longer consists of the non-negative integers, but instead depends on the energy of the state,  $\tilde{a}^\dagger\tilde{a}$  cannot in general be interpreted as the corresponding number operator for nonlinear oscillators. However, like the SHO, states that span  $\mathcal{H}_{nl}$  are enumerated by an integer  $n$ . Thus, we can construct for the nonlinear oscillator an operator  $N(\mathbf{H})$  for which  $N(\mathbf{H})|n\rangle_{nl} = n|n\rangle_{nl}$ . In analogy to the SHO we shall call  $N(\mathbf{H})$  the number operator for the nonlinear oscillator.

We construct this operator by first noting that  $\mathbf{H}$  does not change the occupation number  $n$  of the states  $|n\rangle_{nl}$ . Consequently, the states  $|n\rangle_{nl}$  are simultaneously eigenvectors for both  $\mathbf{H}$  and the constructed number operator. Necessarily, the two operators must commute. Since  $\mathbf{H}$  is the only conserved quantity in the system, the number operator  $N(\mathbf{H})$  can be a function of  $\mathbf{H}$  only. From (15), it is straightforward to see that for  $N(\mathbf{H})|n\rangle_{nl} = n|n\rangle_{nl}$ ,  $N(\mathbf{H})$  must satisfy the commutation relation

$$[\tilde{a}, N(\mathbf{H})] = \tilde{a}. \quad (18)$$

We also require the subsidiary condition  $N(\mathbf{H})|\Omega\rangle_{nl} = 0$ . Expanding  $N(\mathbf{H})$  in a power series and once again using equation (8), the solution to the operator equation (18) reduces to finding the solution of the *functional* equation

$$N(\lambda(e) + e) - N(e) = 1 \quad (19)$$

for a given  $\lambda$  with the 'boundary condition'  $N(e_g) = 0$  (see appendix B).  $e$  is a real number in (19) and  $N(\mathbf{H})$  is obtained by replacing  $e \rightarrow \mathbf{H}$ . (Or, equivalently, (19) is the resultant equation after applying the corresponding operator equation to an eigenstate of  $\mathbf{H}$  with energy  $e$ .) Most physically relevant solutions of equation (19) are monotonically increasing.

Like the differential equation it resembles, the general solution of (19) consists of the linear combination  $N_p(e) + N_h(e)$  where  $N_p(e)$  is the 'particular' solution to (19) while  $N_h(e)$  is the solution to the 'homogeneous' equation

$$N_h(\lambda(e) + e) - N_h(e) = 0. \quad (20)$$

Whether or not such a solution exists depends on the particular form of  $\lambda(e)$ , although the above equations can be solved for general  $\lambda$  in the semiclassical limit as we shall see in

section 5. Unlike a differential equation, however, the single boundary condition  $N(e_g) = 0$  is *not* sufficient to determine  $N(\mathbf{H})$  uniquely in general. Consider the case of the SHO. Then  $\lambda = 1$  and the particular solution of (19) gives  $N_p(\mathbf{H}) = \mathbf{H} - \mathbf{I}/2$ . It is, in fact, the usual number operator. However, the solution of the homogeneous equation (20) is *any periodic function* with period 1 which vanishes at  $e = \frac{1}{2}$ . There are an infinite number of such functions, such as

$$N(\mathbf{H}) = \mathbf{H} - \mathbf{I}/2 + C \sin(\pi(\mathbf{H} - \mathbf{I}/2)) \quad (21)$$

for any real  $C$ .

From equations (5) and (6) the study of nonlinear oscillators reduces to the determination of the ground state energy  $e_g$  and  $\lambda$ . This is non-trivial and a method for doing so will be given in the section 4. For now we shall limit ourselves to a qualitative description of the energy levels by looking at different possible behaviours of  $\lambda(e)$ .

For a ground state to exist,  $\lambda(e) > 0$  and we shall restrict our considerations to such  $\lambda$ . Representing the eigenvalues of  $\mathbf{H}$  generically by  $e$ , if  $\lambda(e)$  is a monotonically increasing function which is unbounded from above, then the energy spacings between successive energy levels becomes wider as  $n$  increases and  $e_n$  grows rapidly with  $n$ . If, on the other hand,  $\lambda(e) \rightarrow \text{constant}$  as  $e \rightarrow \infty$ , then eventually the energy levels become equally spaced and we would once again obtain SHO type of energy levels. Note also, that if we consider (17) as a nonlinear transformation of  $e_n$  generated by  $\lambda(e)$ , then the fixed point of this transformation  $\lambda(\lambda(e) + e) = \lambda(e)$  occurs precisely when  $\lambda(e)$  goes to a constant (see appendix B). Finally, if  $\lambda(e)$  is a monotonically decreasing function of  $e$  which decreases sufficiently rapidly, there will be an upper bound to the energy levels  $e_{\max}$ .

Let us now consider time evolution. If  $\mathbf{H}$  does not explicitly depend on  $t$ , time evolution is generated by a unitary transformation<sup>†</sup>,

$$\tilde{\mathbf{a}}(t) = e^{itH\hbar} \tilde{\mathbf{a}}(0) e^{-itH\hbar} \quad (22)$$

which preserves the commutation relation (6). Using (4), the solution to the Heisenberg equation of motion is

$$\tilde{\mathbf{a}}(t) = e^{-i\lambda(\mathbf{H})t/\hbar} \tilde{\mathbf{a}}(0). \quad (23)$$

The frequency of oscillation of  $\tilde{\mathbf{a}}(t)$ ,  $\lambda(\mathbf{H})/\hbar$ , now depends on the Hamiltonian  $\mathbf{H}$ . This agrees with the recent result of Bender and Bettencourt [13].

The approach taken by Bender and Bettencourt is a generalization of multiple-scale perturbation theory (MSPT) to the solution of quantum evolution equations. It is known that conventional perturbation theory applied to the solution of (classical) differential equations often contain secular terms which grow unphysically with time or length. This is an indication that a characteristic physical behaviour differs at different length scales. MSPT explicitly recognizes this fact and takes it into account. Bender and Bettencourt extended MSPT to the solution of the quantum anharmonic oscillator to first order in the coupling constant and showed that the characteristic frequency of oscillation depends explicitly on the energy of the state. It is interesting to note that the solution to their operator equations automatically diagonalizes their Hamiltonian in the same way we seek solutions to (4) which satisfy (5). However, while their result only holds perturbatively to first order, our result is exact and was obtained algebraically through a completely different approach. The dependence of the frequency of oscillation on the energy of the state was interpreted by them as an operator form of mass renormalization.  $A_n$ 's dependence in (16) on the energy of the state would then be interpreted as wavefunction renormalization.

<sup>†</sup> The case where  $V$  depends on time explicitly, as through the coupling constant  $\epsilon$ , is much more subtle. Neither (22) nor (23) are valid since in general  $[H(t), H(t')] \neq 0$  for  $t \neq t'$ .

### 3. KMS states

In this section we demonstrate the utility of the general results obtained in the above by putting the nonlinear oscillator in contact with a thermal reservoir at a temperature  $T$  and consider the average energy and number operator of the system. We shall denote these thermal averages by  $\langle \cdots \rangle_T$  which we shall take to be a KMS state [27–30]. Namely, if  $\mathbf{A}(t)$  and  $\mathbf{B}(t)$  are two operators in the Heisenberg representation, then

$$\langle \mathbf{A}(t)\mathbf{B}(t) \rangle_T = \langle \mathbf{B}(t)\mathbf{A}(t + i\hbar\beta) \rangle_T \quad (24)$$

where  $1/\beta = k_B T$ . Since thermal equilibrium states are stationary, we can, without a loss of generality, take  $t = 0$  in (24).

Applying this condition to equation (6),

$$\langle \lambda(\mathbf{H}) \rangle_T = \langle \tilde{\mathbf{a}}(0)\tilde{\mathbf{a}}^\dagger(0) \rangle_T - \langle \tilde{\mathbf{a}}(0)\tilde{\mathbf{a}}^\dagger(i\hbar\beta) \rangle_T. \quad (25)$$

Then using the solution, equation (23), of the Heisenberg equation of motion, equation (4) and the commutation relation (6),

$$\langle \lambda(\mathbf{H})e^{-\beta\lambda(\mathbf{H})} \rangle_T = \langle (\mathbf{H} - e_g)(1 - e^{-\beta\lambda(\mathbf{H})}) \rangle_T \quad (26)$$

which reduces to the usual Bose–Einstein distribution for the SHO when  $\lambda = 1$ .

Unlike the case of the SHO, it is not possible to determine  $\langle \mathbf{H} \rangle_T$  any further using solely the KMS condition. We must make use of a partition function and shall restrict ourselves to states which can be represented by a trace over a density matrix,

$$\langle \mathbf{H} \rangle_T = \frac{1}{Z} \text{Tr}_{\mathcal{H}_{nl}} \mathbf{H} e^{-\beta\mathbf{H}} \quad (27)$$

where  $Z \equiv \text{Tr}_{\mathcal{H}_{nl}} e^{-\beta\mathbf{H}}$  is the usual partition function. Then using the identity

$$\langle [\mathbf{H} + \lambda(\mathbf{H})]e^{-\beta\lambda(\mathbf{H})} \rangle_T = \langle e^{-\beta\lambda(\mathbf{H})} \rangle_T \langle \mathbf{H} \rangle_T - \frac{\partial}{\partial\beta} \langle e^{-\beta\lambda(\mathbf{H})} \rangle_T \quad (28)$$

we obtain

$$\langle \mathbf{H} \rangle_T = e_g - \frac{\frac{\partial}{\partial\beta} \langle e^{-\beta\lambda(\mathbf{H})} \rangle_T}{1 - \langle e^{-\beta\lambda(\mathbf{H})} \rangle_T} \quad (29)$$

and we see once again the importance of  $\lambda(\mathbf{H})$ . Indeed, from (27) we find that

$$Z = \frac{e^{-\beta e_g}}{1 - \langle e^{-\beta\lambda(\mathbf{H})} \rangle_T}. \quad (30)$$

As for the number operator, from equation (19),

$$\langle e^{-\beta\lambda(\mathbf{H})} \rangle_T = \langle N(\mathbf{H} + \lambda(\mathbf{H}))e^{-\beta\lambda(\mathbf{H})} \rangle_T - \langle N(\mathbf{H})e^{-\beta\lambda(\mathbf{H})} \rangle_T. \quad (31)$$

Then, using

$$\langle N(\mathbf{H} + \lambda(\mathbf{H}))e^{-\beta\lambda(\mathbf{H})} \rangle_T = \frac{1}{Z} \text{Tr}_{\mathcal{H}_{nl}} N(\mathbf{H} + \lambda(\mathbf{H}))e^{-\beta\mathbf{H} - \beta\lambda(\mathbf{H})} \quad (32)$$

and (17), we find that  $\langle N(\mathbf{H} + \lambda(\mathbf{H}))e^{-\beta\lambda(\mathbf{H})} \rangle_T = \langle N(\mathbf{H}) \rangle_T$  so that

$$\langle e^{-\beta\lambda(\mathbf{H})} \rangle_T = \langle N(\mathbf{H})(1 - e^{-\beta\lambda(\mathbf{H})}) \rangle_T. \quad (33)$$

This once again agrees with the SHO result for  $\lambda = 1$ .



**4. Solution of the eigenvalue problem**

In the previous section we have analysed the general structure of the Hilbert spaces of nonlinear oscillators and their properties when placed in contact with a heat bath. Our approach depends on the solution of the eigenvalue problem equations (4) and (5). In this section we shall present a method of constructing these solutions  $\lambda(\mathbf{H})$ ,  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{a}}^\dagger$ . Since  $\mathbf{H}$  is given in terms of  $\mathbf{a}$  and  $\mathbf{a}^\dagger$ , in general  $\tilde{\mathbf{a}} = \tilde{\mathbf{a}}(\mathbf{a}, \mathbf{a}^\dagger)$ , which is understood in terms of a power series,

$$\tilde{\mathbf{a}} = \sum_{r,s=0}^{\infty} b_{rs} (\mathbf{a}^\dagger)^r \mathbf{a}^s \tag{34}$$

By using the commutation relation  $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{I}$  we can always reduce any expansion of  $\tilde{\mathbf{a}}$  to this form. Equation (34) is well defined only if the corresponding function

$$f(z, \bar{z}) = \sum_{r,s=0}^{\infty} b_{rs} \bar{z}^r z^s \tag{35}$$

is convergent on the complex plane  $\mathcal{C}$ .

At this point we should also express  $\lambda(\mathbf{H})$  as a power series in  $\mathbf{H}$ , insert this series and as well as (34) in equation (4) and obtain an infinite set of coupled equations between various  $b_{rs}$  and the coefficients of the  $\lambda$  expansion. The problem would quickly become intractable, however. We shall therefore first make the following drastic simplification. Instead of (4) we shall solve the simpler equation

$$: [\tilde{\mathbf{a}}, \mathbf{H}] :=: \lambda^{(sc)}(\mathbf{H}) \tilde{\mathbf{a}} : \tag{36}$$

where  $:=:$  denotes normal ordering. Since the solution of (36) will differ from that of (4), we denote the left eigenvalue of (36) by  $\lambda^{(sc)}$ . Indeed, as we shall see, once this is done we are in the semiclassical limit. Correspondingly, we shall take the normalization condition as

$$: \mathbf{H} :=: \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} : + \frac{1}{2} \mathbf{I}. \tag{37}$$

The solution of this equation is then used as a guide to reconstructing the solution to (4). Notice that corrections to the ground state energy cannot be determined under this simplification. It can only be determined when the full operators are reconstructed from the solution to (36).

Denoting the solution to (36) by the superscript  $sc$ , we find that for

$$\tilde{\mathbf{a}}^{(sc)} = \sum_{r,s=0}^{\infty} b_{rs}^{(sc)} (\mathbf{a}^\dagger)^r \mathbf{a}^s \tag{38}$$

we have

$$\begin{aligned} : \sum_{r,s=0}^{\infty} b_{rs}^{(sc)} \lambda(\mathbf{H}) (\mathbf{a}^\dagger)^r \mathbf{a}^s :=: & \sum_{r,s=0}^{\infty} b_{rs}^{(sc)} \{ (s-r) (\mathbf{a}^\dagger)^r \mathbf{a}^s + s : (\mathbf{a}^\dagger)^r \mathbf{a}^{s-1} [\mathbf{a}, V(\mathbf{a}, \mathbf{a}^\dagger)] : \\ & + r : (\mathbf{a}^\dagger)^{r-1} [\mathbf{a}^\dagger, V(\mathbf{a}, \mathbf{a}^\dagger)] \mathbf{a}^s : \}. \end{aligned} \tag{39}$$

Under this normal ordering, solving (39) is equivalent to solving the differential equation

$$\lambda^{(sc)}(e^{(sc)}) f^{(sc)}(z, \bar{z}) = \{ f^{(sc)}, e^{(sc)} \}_{PB} \equiv \frac{\partial f^{(sc)}}{\partial z} \frac{\partial e^{(sc)}}{\partial \bar{z}} - \frac{\partial f^{(sc)}}{\partial \bar{z}} \frac{\partial e^{(sc)}}{\partial z} \tag{40}$$

where  $\tilde{\mathbf{a}}^{(sc)} = f^{(sc)}(\mathbf{a}, \mathbf{a}^\dagger)$ .  $e^{(sc)}$  is obtained from  $\mathbf{H}$  by replacing everywhere  $\mathbf{a} \rightarrow z$  and  $\mathbf{a}^\dagger \rightarrow \bar{z}$  and the normalization condition (37) is now  $e^{(sc)} = |f^{(sc)}|^2 + \frac{1}{2}$ . The right-hand side of (40) is just the classical Poisson bracket but with the generalized coordinates

$$z = \left( \frac{m}{2\hbar^2} \right)^{1/2} x + i \left( \frac{1}{2m} \right)^{1/2} p \quad \bar{z} = \left( \frac{m}{2\hbar^2} \right)^{1/2} x - i \left( \frac{1}{2m} \right)^{1/2} p \tag{41}$$

where  $m$  is the mass of the particle. We are therefore looking for a semiclassical solution to (5). Indeed, we shall see explicitly in section 5 that the solution of (40) is equivalent to the WKB approximation.

Importantly, (40) has the same symmetry properties as (4). Namely, if  $f^{(sc)}(z, \bar{z})$  is a solution to (40), then so is  $f^{(sc)}(z, \bar{z})g(e^{(sc)})$  where  $g(e^{(sc)})$  is any function of  $e^{(sc)}$  (although they do not satisfy the normalization condition equation (37)). Making use of this symmetry, we change coordinates from  $|z|$  and  $\theta = -i \log(z/\bar{z})/2$  to  $e^{(sc)}$  and  $\theta$  so that

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial e} \frac{\partial e}{\partial z} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial z} \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial e} \frac{\partial e}{\partial \bar{z}} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial \bar{z}}. \quad (42)$$

Terms containing derivatives of  $e$  vanish and (40) reduces to

$$i\lambda^{(sc)}(e^{(sc)})f^{(sc)}(e^{(sc)}, \theta) = \frac{\partial e^{(sc)}}{\partial |z|^2} \frac{\partial f^{(sc)}}{\partial \theta} \quad (43)$$

whose solution is

$$f^{(sc)}(e^{(sc)}, \theta) = \sqrt{e^{(sc)} - \frac{1}{2}} \exp \left\{ i\lambda^{(sc)} \int_0^\theta \left( \frac{\partial e^{(sc)}}{\partial |z|^2} \right)^{-1} d\phi \right\} \quad (44)$$

and satisfies the normalization condition explicitly. Determination of  $\tilde{a}^{(sc)}$  is then reduced to performing the above integral, which requires inverting the equation  $e^{(sc)} = e^{(sc)}(|z|, \theta)$  and solving for  $|z|$  in terms of  $e^{(sc)}$  and  $\theta$ . Next, for  $f^{(sc)}$  to be analytic on  $\mathcal{C}$ ,  $f^{(sc)}(e^{(sc)}, 0) = f^{(sc)}(e^{(sc)}, 2\pi)$ , giving

$$\frac{1}{\lambda^{(sc)}(e^{(sc)})} = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\partial e^{(sc)}}{\partial |z|^2} \right)^{-1} d\phi \quad (45)$$

which determines  $\lambda^{(sc)}$ . Notice that in contrast to phase angle techniques which require the construction of a phase operator (see, for example, [22]) and its concomitant difficulties, our analysis uses the phase only in the semiclassical limit where it is well defined.

Reconstruction of  $\tilde{a}$  and  $\lambda(\mathbf{H})$  from  $f^{(sc)}(z, \bar{z})$  is now straightforward, although tedious.  $\tilde{a}^{(sc)}$  can be obtained by first expanding  $f^{(sc)}$  in (44) in a power series in  $z$  and  $\bar{z}$ , then taking  $\tilde{a}^{(sc)} = f^{(sc)}(\mathbf{a}, \mathbf{a}^\dagger)$ . Since  $\tilde{a}^{(sc)}$  was obtained via normal ordering, there is an ordering ambiguity when we reconstruct  $\tilde{a}$  from it. Fundamentally, this arises when we replace  $z \rightarrow \mathbf{a}$ ,  $\bar{z} \rightarrow \mathbf{a}^\dagger$  in  $f^{(sc)}(\mathbf{a}, \mathbf{a}^\dagger)$  since the term  $\bar{z}z$  in the expansion can be replaced by either  $\mathbf{a}^\dagger \mathbf{a}$  or  $\mathbf{a} \mathbf{a}^\dagger = \mathbf{I} + \mathbf{a}^\dagger \mathbf{a}$ . Therefore, to determine  $\tilde{a}$  we shall take  $\tilde{\mathbf{a}} = \{f^{(sc)}(\mathbf{a}, \mathbf{a}^\dagger)\}_{\text{order}}$ , but we now replace  $\bar{z}z \rightarrow \mathbf{a}^\dagger \mathbf{a} + A\mathbf{I}$  where the  $A$ 's are constants. These are determined by requiring that the resulting expansions for  $\tilde{a}$  and  $\lambda(\mathbf{H})$  satisfy both (4) and (5) (or equivalently (6)) term by term in the expansion. This uniquely determines not only  $\tilde{a}$  and  $\lambda(\mathbf{H})$ , but  $e_g$  as well.

From (45) we see that for  $\lambda(\mathbf{H})$  to be independent of  $\mathbf{H}$ ,  $\partial e^{(sc)}/\partial |z|^2 = k$ , where  $k$  is a function of  $\theta$  only. This limits  $l \leq 2$ . Correspondingly, if  $l > 2$ ,  $\lambda(\mathbf{H})$  is necessarily a function of  $\mathbf{H}$ .

## 5. The WKB approximation

We now make the connection between the solution of (40) and the semiclassical limit. From the correspondence principle, in the large- $n$  limit  $e_n$  goes over to the classical result in which the spacings between energy levels  $e_n - e_{n-1}$  are small in comparison to  $e_{n-1}$  and the levels are essentially continuous. In this limit, we can approximate

$$e_n - e_{n-1} \approx \frac{de}{dn} \quad (46)$$

where  $e(n)$  is considered as a continuous function of  $n$ . Then from (17)

$$\frac{de}{dn} \approx \lambda^{(sc)}(e) \tag{47}$$

where we have replaced  $\lambda \rightarrow \lambda^{(sc)}$  in this limit. Integrating and using equations (41) and (45),

$$n + n_\infty \approx \frac{1}{2\pi} \int_0^e \int_0^{2\pi} \left( \frac{\partial e^{(sc)}}{\partial |z|^2} \right)^{-1} de^{(sc)} d\theta \tag{48}$$

where  $n_\infty$  is an integration constant which can be neglected in the limit  $n \rightarrow \infty$ . Changing variables back to  $z$  and  $\bar{z}$  in the integrand of (48) and using (41), we find that

$$n + n_\infty \approx \frac{1}{2\pi\hbar} \iint_{\mathcal{D}_e} dx dp. \tag{49}$$

The integration is now over a disc  $\mathcal{D}_e$  centred about the origin in the classical phase space. This is just the semiclassical Bohr–Sommerfeld quantization rule obtained from the WKB approximation.

We next consider the solution of the functional equation (19) in the large- $n$  limit. Since  $\lambda(e)$  measures the energy splitting between energy levels, in this limit  $\lambda(e) \ll e$  (see appendix A) and (16) can be approximated by the differential equation

$$\lambda^{(sc)}(e) \frac{dN}{de} \approx 1 \tag{50}$$

where once again we have replaced  $\lambda \rightarrow \lambda^{(sc)}$ . The solution to this equation is trivial and we once again obtain the WKB result,

$$N(e) + n_\infty \approx \frac{1}{2\pi\hbar} \iint_{\mathcal{D}_e} dx dp \tag{51}$$

in the semiclassical limit. However, the operator  $N(\mathbf{H})$  can now be obtained directly from (51) by expanding the integral in powers of  $e$  and replacing  $e \rightarrow \mathbf{H}$ .

Finally, we consider the quantum partition function

$$Z \equiv \text{Tr}_{\mathcal{H}_{nl}} e^{-\beta\mathbf{H}} = \sum_{n=0}^{\infty} e^{-\beta e_n} \tag{52}$$

in the large-temperature limit. Making use of the Euler–Maclaurin formula,

$$Z \approx \int_0^{\infty} e^{-\beta e} dn + \frac{1}{2} e^{-\beta e_s} + \text{H.O.T.} \tag{53}$$

In the large-temperature limit  $k_B T \gg e_g$  we can neglect the terms  $\sim e^{-\beta e_s}$ . Moreover, at this energy scale,  $e_n \gg \lambda(e_n)$ . Using (47), we convert the integral over  $n$  to one over  $e$ . Making use once again of (41), we find that

$$Z \approx \frac{1}{2\pi\hbar} \iint e^{-\beta e} dx dp \tag{54}$$

where the integral is over the classical phase space. This is precisely the classical result with the requisite factor of the fundamental phase space volume  $2\pi\hbar$ .

### 6. The $x^4$ interaction

In this section we shall apply the above analysis to a non-trivial system, the  $x^4$  anharmonic oscillator. As an example of the above techniques, we shall explicitly construct  $\lambda$ ,  $\tilde{a}$  and  $\tilde{a}^\dagger$ . The Hamiltonian for the oscillator is

$$\mathbf{H} = \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \mathbf{I} + \frac{\epsilon}{4} (\mathbf{a} + \mathbf{a}^\dagger)^4 \tag{55}$$

with the corresponding

$$e^{(sc)} = \frac{1}{2} + |z|^2 + 4\epsilon|z|^4 \cos^4 \theta. \quad (56)$$

Then

$$f^{(sc)}(e^{(sc)}, \theta) = \sqrt{e^{(sc)} - \frac{1}{2}} \exp \left\{ \frac{\pi i}{2} \frac{I(\theta)}{I(\pi/2)} \right\} \quad (57)$$

where

$$I(\theta) = \int_0^\theta \frac{d\phi}{\sqrt{1 + \xi \cos^4 \phi}} \quad (58)$$

and  $\xi = 16(e^{(sc)} - \frac{1}{2})\epsilon$ . When  $\epsilon > 0$ , this integral can be reduced to

$$I(\theta) = \frac{1}{2(1 + \xi)^{1/4}} F(\alpha(\theta)|q) \quad (59)$$

where  $F(\alpha(\theta)|q)$  is the elliptical integral of the first kind and

$$\alpha(\theta) = \arccos \left( \frac{\sqrt{1 + \xi} - \tan^2 \theta}{\sqrt{1 + \xi} + \tan^2 \theta} \right) \quad (60)$$

while

$$q = \frac{\sqrt{1 + \xi} - 1}{2\sqrt{1 + \xi}} \quad (61)$$

is its modulus. The analyticity of  $f^{(sc)}(e^{(sc)}, \theta)$  gives

$$\lambda^{(sc)}(e^{(sc)}) = \frac{\pi}{2} \frac{(1 + \xi)^{1/4}}{K(\sqrt{q})} \quad (62)$$

where  $K(\sqrt{q})$  is the quarter-period of  $F(\alpha(\theta)|q)$ .

Next, to determine  $\tilde{\mathbf{a}}$ , we expand (57) to fifth order in  $|z|$ ,

$$f^{(sc)}(z, \bar{z}) = z + \frac{1}{4}\epsilon \{-3(z^2 - \bar{z}^2)z + (z + \bar{z})^3\} + \frac{1}{2}\epsilon^2 \left\{ \frac{3}{2}z^5 + \frac{39}{4}\bar{z}z^4 - \frac{25}{8}\bar{z}^2z^3 - 12\bar{z}^3z^2 - \frac{3}{8}\bar{z}^4z + \frac{1}{4}\bar{z}^5 \right\}. \quad (63)$$

Notice that this is an expansion in  $|z|$ , whose convergence is guaranteed by the analyticity of  $f^{(sc)}$ , and *not* in the coupling constant  $\epsilon$ . We then replace  $z \rightarrow \mathbf{a}$  and  $\bar{z} \rightarrow \mathbf{a}^\dagger$  in the above and take

$$\tilde{\mathbf{a}} = \mathbf{a} + \frac{1}{4}\epsilon \mathbf{F} + \frac{1}{2}\epsilon^2 \mathbf{G} \quad (64)$$

where

$$\begin{aligned} \mathbf{F} &= -3(\mathbf{a}^2 - (\mathbf{a}^\dagger)^2)\mathbf{a} + (\mathbf{a} + \mathbf{a}^\dagger)^3 + f_1\mathbf{a} + f_2\mathbf{a}^\dagger \\ \mathbf{G} &= \frac{3}{2}\mathbf{a}^5 + \frac{39}{4}\mathbf{a}^\dagger\mathbf{a}^4 - \frac{25}{8}(\mathbf{a}^\dagger)^2\mathbf{a}^3 - 12(\mathbf{a}^\dagger)^3\mathbf{a}^2 - \frac{3}{8}(\mathbf{a}^\dagger)^4\mathbf{a} + \frac{1}{4}(\mathbf{a}^\dagger)^5 \\ &\quad + g_1\mathbf{a}^3 + g_2\mathbf{a}^\dagger\mathbf{a}^2 + g_3(\mathbf{a}^\dagger)^2\mathbf{a} + g_4(\mathbf{a}^\dagger)^3 + g_5\mathbf{a} + g_6\mathbf{a}^\dagger. \end{aligned} \quad (65)$$

The constants  $f_1, f_2, g_1, \dots, g_6$  are present due to the ordering ambiguity. Requiring that (64) satisfies (4) gives  $f_1 = f_2 = 3$ , while

$$g_1 + g_3 = -15 \quad g_2 = -\frac{135}{8} \quad g_4 = -\frac{3}{8} \quad g_5 = -\frac{153}{8} \quad g_6 = -\frac{27}{2}. \quad (66)$$

The ground state energy of the oscillator is also determined to this order,

$$e_g = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2. \quad (67)$$

Now using the commutation relation (6), we obtain

$$\lambda(\mathbf{H}) = \mathbf{I} + 3\epsilon(\mathbf{H} + \frac{1}{2}\mathbf{I}) - \epsilon^2 \left\{ \frac{69}{4}(\mathbf{H} + \frac{1}{2}\mathbf{I})^2 - \frac{9}{2}(\mathbf{H} + \frac{1}{2}\mathbf{I}) + \frac{15}{2}\mathbf{I} \right\} \quad (68)$$

while

$$3g_1 + g_3 = \frac{45}{2} \tag{69}$$

giving  $g_1 = \frac{75}{4}$  and  $g_3 = -\frac{135}{4}$ . This last relationship was obtained by requiring that  $\lambda$  be a function of  $H$  only. To this order then,

$$\begin{aligned} \tilde{a} = a + \frac{1}{4}\epsilon \{ & -3(a^2 - (a^\dagger)^2)a + (a + a^\dagger)^3 + 3(a + a^\dagger) \} \\ & + \frac{1}{2}\epsilon^2 \{ \frac{3}{2}a^5 + \frac{39}{4}a^\dagger a^4 - \frac{25}{8}(a^\dagger)^2 a^3 - 12(a^\dagger)^3 a^2 - \frac{3}{8}(a^\dagger)^4 a + \frac{1}{4}(a^\dagger)^5 \\ & + \frac{75}{4}a^3 - \frac{135}{8}a^\dagger a^2 - \frac{135}{4}(a^\dagger)^2 a - \frac{3}{8}(a^\dagger)^3 - \frac{153}{8}a - \frac{27}{2}a^\dagger \}. \end{aligned} \tag{70}$$

The energy levels can now be straightforwardly calculated from (17),

$$\begin{aligned} e_n &= e_{n-1} + \lambda(e_{n-1}) \\ &= e_g + \sum_{r=0}^{n-1} \lambda(e_r). \end{aligned} \tag{71}$$

Using (68), and keeping terms to order  $\epsilon^2$  only, we obtain after rearrangement,

$$e_n \approx e_g + n + 3\epsilon \sum_{r=0}^{n-1} (r+1) - \epsilon^2 \left( \frac{51}{4} \sum_{r=0}^{n-1} r^2 + \frac{51}{2} \sum_{r=0}^{n-1} r + 18n + \frac{21}{8} \right). \tag{72}$$

Then

$$e_n \approx n + \frac{1}{2} + \frac{3}{4}\epsilon(2n^2 + 2n + 1) - \epsilon^2 \left( \frac{17}{4}n^3 + \frac{51}{8}n^2 + \frac{59}{8}n + \frac{21}{8} \right) \tag{73}$$

which is the standard second-order perturbation result. Notice also that if we keep terms only up to  $\epsilon$ , then  $e_n - e_{n-1} \equiv \lambda(e_n) \approx 1 + 3n\epsilon$ . This is precisely the result obtained by Bender and Bettencourt [13].

It is instructive to compare (68) with the expansion of (62),

$$\lambda^{(sc)}(e) = 1 + 3\epsilon(e^{(sc)} - \frac{1}{2}) - \frac{69}{4}\epsilon^2(e^{(sc)} - \frac{1}{2})^2. \tag{74}$$

Notice that in both expansions the coefficients of the highest power of the energy in each term are the same. This is a generic feature. Quantum mechanical corrections to  $\lambda^{(sc)}$  only results in the appearance of lower powers of  $H$  in each term of the expansion. Moreover, if we then use  $\lambda^{(sc)}$  to calculate  $e_n$ , we find

$$e_n^{(sc)} = n + \frac{1}{2} + \frac{3}{2}\epsilon(n^2 - n) - \epsilon^2 \left( \frac{17}{4}n^3 - \frac{33}{8}n^2 - \frac{1}{8}n \right) \tag{75}$$

which also agrees with (71) in the large- $n$  limit. This also is a generic feature of the expansion since the coefficient of the highest power of  $n$  in each term of the expansion is obtained from  $\lambda^{(sc)}$  only.

The above result is valid only for small  $n$ . In the large- $n$  limit, the semiclassical result is valid and

$$\lambda(e_n) \approx \lambda^{(sc)}(e_n) \approx \frac{\pi}{K(1/\sqrt{2})} (e\epsilon)^{1/4}. \tag{76}$$

To compare with the WKB result, from [7] we know that

$$e_n^{WKB} \approx \frac{3^{4/3}\pi^2}{[\Gamma(\frac{1}{4})]^{8/3}} \epsilon^{1/3} n^{4/3}. \tag{77}$$

This gives the energy splitting between levels as

$$e_{n+1}^{WKB} - e_n^{WKB} \approx \frac{4}{3} \frac{e^{WKB}}{n} \approx \frac{4\pi^{3/2}}{[\Gamma(\frac{1}{4})]^2} (e_n^{WKB} \epsilon)^{1/4}. \tag{78}$$

Since  $K(1/\sqrt{2}) = [\Gamma(\frac{1}{4})]^2/4/\sqrt{\pi}$ , this is precisely the form of  $\lambda(e_n)$  for large  $n$  and we see explicitly the equivalence between  $\lambda^{(sc)}$  and the WKB approximation.

## 7. Discussion

We have shown that the study of nonlinear oscillators is equivalent to the study of algebras satisfying equations (5) and (6), the SHO being a special case of this algebra. The Hilbert space (15) and eigenvalues (17) of these algebras all have the same form. The number operator for nonlinear oscillators was also constructed. Results of this general analysis were used to determine the general form of the partition function and average energy for a nonlinear oscillator in contact with a heat bath. Analysis of nonlinear oscillators therefore reduces to determining the function  $\lambda(\mathbf{H})$  and the ground state  $e_g$  of the oscillator. This can be done by first making a semiclassical approximation, which requires only the evaluation of a single integral, and then using it as a guide to constructing  $\tilde{a}$  and  $\tilde{a}^\dagger$  in terms of  $a$  and  $a^\dagger$ .

Unlike the Bogoliubov transformation, the mapping between  $(\tilde{a}, \tilde{a}^\dagger)$  and  $(a, a^\dagger)$  is nonlinear and cannot be generated by a simple unitary transformation. The two Hilbert spaces  $\mathcal{H}_\epsilon$  and  $\mathcal{H}_{SHO}$  are unitarily inequivalent. Indeed, each value of  $\epsilon$  determines a separate Hilbert space, all of whom are inequivalent to one another. This result provides a concrete example of Haag's theorem first proved for quantum field theories in higher dimensions. Based on the results of this theorem and the generality of our analysis, we expect a similar construction to hold for the  $\phi^4$  theory in higher dimensions. Note, however, that this construction requires a natural energy scale to define  $\lambda(\mathbf{H})$ . For Hamiltonians of the form of equation (1) the SHO energy scale plays that role. For quantum field theories, however, no such natural energy scale exists. It would have to be introduced, requiring an introduction of a high (or low) energy cutoff for the theory.

Once the general solution to (4) is known, the general structure of the Hilbert space and the eigenvalues of nonlinear oscillators is determined. The establishment of the existence of such solutions for general  $\mathbf{H}$  is quite difficult. However, we have shown the utility of this approach by applying it to the general analysis of the quantum statistical mechanical properties of nonlinear oscillators in contact with a heat bath, and we have shown that they reduce to the standard result for the case of the SHO. We have also presented a specific method of solving these equations and have shown that in the semiclassical limit it reduces to the WKB result for general  $\mathbf{H}$ . Consequently,  $\lambda$ ,  $\tilde{a}$  and  $\tilde{a}^\dagger$  exist in this limit. Indeed, the partition function for nonlinear oscillators reduces to the usual integral over the classical phase space in this limit. In the full quantum mechanical case, we have shown that in the special case of the  $x^4$  oscillator, not only does our method of constructing solutions of (4) agree explicitly with the WKB result, it also agrees with the results obtained via standard perturbation theory to the second order. Just as importantly, it agrees with the results of Bender and Bettencourt obtained using quantum MSPT, a completely independent method.

The method of finding solutions to the nonlinear oscillator problem given in sections 2 and 4 does not make explicit use of the states of any specific Hilbert space. This was deliberate and needed in light of the fact that  $\mathcal{H}_{nl}$  and  $\mathcal{H}_{SHO}$  are unitarily inequivalent. Rather, we solve this problem by explicitly constructing the operators which generate the eigenstates of the nonlinear oscillator from  $a$  and  $a^\dagger$ . This is in contrast to the usual perturbation theory approach in which one begins with  $\mathcal{H}_{SHO}$  from the very beginning and construct the energy eigenstates of the nonlinear oscillator from those of the SHO. While the approach taken in sections 4 and 6 would seem to be more tedious than the usual perturbation theory, it is known that the perturbative result for the case of the anharmonic oscillator does not converge in the traditional sense and needs to be Borel summed to obtain a finite result. It is important to note that the expansion of  $\tilde{a}$  in (34) does *not* rely on an expansion in the coupling constant  $\epsilon$  as perturbation theory does. Rather it is an expansion in  $a$  and  $a^\dagger$ , as can be explicitly seen in equations (63) and (70) for the anharmonic oscillator. In the semiclassical limit the

convergence of this series can be determined by studying the analytical properties of (44). Indeed, convergence of (34) is guaranteed for the anharmonic oscillator in this limit since for this case  $f^{(sc)}$  is simply related to the elliptical functions whose analytic properties are well known. While analyticity in the semiclassical limit does not guarantee the convergence of (34), failure of convergence in this limit virtually assures that (34) will not converge. Nevertheless, it is still an open question as to the classes of potentials for which (34) converges.

### Appendix A

The existence of a ground state is only guaranteed when  $\lambda$  is positive definite. In the semiclassical limit it can be shown that this holds for wide classes of bounding potentials. For the polynomial interaction potential  $V(a, a^\dagger) = \epsilon(a + a^\dagger)^l/l$ ,

$$\lambda^{(sc)} \sim e^{1/2-1/l} \epsilon^{1/l}. \quad (79)$$

For the exponential interaction potential  $V(a, a^\dagger) = \epsilon e^{\alpha^2(a+a^\dagger)^2}$ ,

$$\lambda^{(sc)} \sim \alpha \sqrt{\frac{e}{\log(e/\epsilon)}}. \quad (80)$$

In both cases  $\lambda^{(sc)}$  is positive definite. Notice also that  $\lambda^{(sc)}/e^{(sc)} \rightarrow 0$  as  $e^{(sc)} \rightarrow \infty$ , justifying the approximations made in section 5.

### Appendix B

The form of (8), as well as the functional equation (19), suggest that we look at a finite-difference form of the one-dimensional Lie derivative. Given any function  $\lambda(x)$ , we define the finite-difference Lie operator  $\mathcal{L}_\lambda$  by

$$\mathcal{L}_\lambda f(x) \equiv f(\lambda(x) + x) - f(x) \quad (81)$$

where  $f(x)$  is any function of  $x$ . In the limit  $\lambda(x) \rightarrow 0$  for all  $x$ , it is straightforward to see that (81) reduces to the usual Lie derivative. Moreover, for any two functions  $f(x)$  and  $g(x)$  and constants  $a, b$ ,

$$\mathcal{L}_\lambda(af(x) + bg(x)) = a\mathcal{L}_\lambda f(x) + b\mathcal{L}_\lambda g(x). \quad (82)$$

$\mathcal{L}_\lambda$  is therefore a linear operator on the space of functions on  $\mathcal{R}$ . It is *not*, however, a derivation since it does not satisfy the product rule,

$$\mathcal{L}_\lambda\{f(x)g(x)\} = f(x)\mathcal{L}_\lambda g(x) + g(x)\mathcal{L}_\lambda f(x) + \mathcal{L}_\lambda f(x)\mathcal{L}_\lambda g(x). \quad (83)$$

Finally, for any two given functions  $\lambda(x)$  and  $\xi(x)$ , the commutator of two finite-difference Lie operators

$$[\mathcal{L}_\lambda, \mathcal{L}_\xi]f(x) = f(x + \lambda(x) + \xi(x + \lambda(x))) - f(x + \xi(x) + \lambda(x + \xi(x))) \quad (84)$$

vanishes if and only if  $\mathcal{L}_\lambda \xi(x) = \mathcal{L}_\xi \lambda(x)$ .

Using  $\mathcal{L}_\lambda$ , the particular solution of (19) becomes the solution of the operator equation,  $\mathcal{L}_\lambda N_p = 1$ . Of more interest is the homogeneous solution to (19),  $\mathcal{L}_\lambda N_h = 0$ , which lies in the kernel of  $\mathcal{L}_\lambda$ ,  $\ker \mathcal{L}_\lambda$ , for a given  $\lambda$ . Notice that when  $\lambda$  is a real constant,  $\ker \mathcal{L}_\lambda$  is the space of all periodic functions with period  $1/\lambda$ . When  $\lambda$  is a function of  $x$ ,  $\ker \mathcal{L}_\lambda$  contains generalizations of periodic functions to those whose frequencies are  $x$ -dependent. This agrees quite well with the observation that in the semiclassical limit,  $1/\lambda^{(sc)}$  reproduces the WKB result. Of particular interest is when  $\lambda \in \ker \mathcal{L}_\lambda$ :  $\mathcal{L}_\lambda \lambda = 0$ . From (17) we see that for this  $\lambda$  the energy levels are equally spaced and are determined solely by  $\lambda(e_g)$ . The constant function, and thus the SHO, clearly satisfies this condition. Whether there are other non-trivial solutions to this equation for physically realizable nonlinear oscillators is still an open question.

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